# 7

# BOUNDARY CONDITIONS

The colliding plane wave problem has been formulated in the previous chapters as a characteristic initial value problem with initial data specified on two null hypersurfaces. At this point it is necessary to discuss the junction conditions between the four regions described in Figure 3.1, and particularly on the initial boundaries of region IV.

## 7.1 General discussion

When pasting together particular solutions of Einstein's equations applying to different regions of space-time, it has generally been thought that the appropriate junction conditions are those of Lichnerowicz (1955). These require that there exist coordinates in which the metric tensor is $C^1$ and piecewise at least $C^2$. These conditions appear to be reasonable since the curvature tensor involves the second derivatives of the metric tensor.

Work on junction conditions up to 1966 has been surveyed by Israel (1966), with particular emphasis on conditions across non-null boundaries. He has also shown that, if the distributional part of the energy-momentum tensor is zero, then the second fundamental forms induced by the metric match across the boundary, and this implies that the full Riemann tensor is regular. This condition was proposed by Darmois (1927), and has been shown to be equivalent to the Lichnerowicz conditions by Bonnor and Vickers (1981).

An alternative set of conditions has been proposed by O'Brien and Synge (1952). In the non-null case, these also have been shown by Israel (1958) and Robson (1972) to be equivalent to those of Lichnerowicz. However, since these are not stated covariantly, some differences may occur as discussed by Bonnor and Vickers (1981).

O'Brien and Synge (1952) have also proposed conditions across null boundaries that are weaker than those of Lichnerowicz. Denoting the null hypersurface by $x^0 =$ constant, with $g_{00} = 0$, they propose simply that the components

$$g_{\mu\nu}, \qquad g^{ij}g_{ij,0}, \qquad g^{i0}g_{ij,0} \tag{7.1}$$

where $(i, j = 1, 2, 3)$, should be continuous across $x^0 =$ constant.

By requiring that the metric tensor be $C^1$ and piecewise at least $C^2$, the Lichnerowicz conditions specifically exclude the impulsive gravitational waves that are discussed in Chapter 3. It can be shown, however, that the line element (3.3) does satisfy the O'Brien–Synge conditions across the null hypersurface $u=0$. This connects the metrics (3.6) and (3.7) in regions I and II. The importance of an appropriate choice of coordinates may also be pointed out here, since, when the continuous line element (3.3) is transformed to the form (3.1), the metric tensor then contains a delta function discontinuity.

It has been shown by Robson (1973), that the appropriate junction conditions across a null hypersurface are those of O'Brien and Synge (1952). Junction conditions across null hypersurfaces have been further analysed by Penrose (1972) and Clarke and Dray (1987). In addition, Bell and Szekeres (1974) have shown that, for colliding plane electromagnetic waves, the Lichnerowicz conditions have to be relaxed in favour of those of O'Brien and Synge.

## 7.2 Junction conditions for colliding plane waves

In order to discuss the collision of plane waves, it has been found convenient to divide space-time up into four regions as described in Figure 3.1. These regions are bounded by the two null hypersurfaces $u=0$ and $v=0$. It has also been suggested in Section 6.4, that it is convenient to assume that the line element (6.20) applies to the entire space-time. However, the metric functions $U$, $V$, $W$ and $M$ must take different forms in the four regions.

It is now necessary to consider the conditions that should be imposed on these functions at the boundaries of the four regions. The O'Brien–Synge conditions (7.1) imply that $V$, $W$ and $M$ are continuous and that $U$ is smooth across these null boundaries. However, according to (6.24),

$$U = -\log(f(u)+g(v)). \tag{7.2}$$

It is therefore necessary that $f(u)$ and $g(v)$ are at least $C^1$.

In order for the line element (6.20) to describe a collision of plane waves, it must be assumed that $U$, $V$, $W$ and $M$ are functions of $u$ and $v$ in region IV, are functions of $u$ only in region II, are functions of $v$ only in region III, and are constants in region I. Concentrating initially on the metric function $e^{-U}=f+g$, it is appropriate to choose

$$\begin{aligned} f &= \tfrac{1}{2} \qquad \text{for} \;\; u \le 0, \qquad f'(0)=0 \\ g &= \tfrac{1}{2} \qquad \text{for} \;\; v \le 0, \qquad g'(0)=0. \end{aligned} \tag{7.3}$$

The approaching waves are then partly described by $f(u)$ in region II, and by $g(v)$ in region III. With this choice it may now be noted that

$$\begin{aligned} &\text{in region I}: && e^{-U} = 1 \\ &\text{in region II}: && e^{-U} = \tfrac{1}{2} + f(u) \\ &\text{in region III}: && e^{-U} = \tfrac{1}{2} + g(v) \\ &\text{in region IV}: && e^{-U} = f(u) + g(v). \end{aligned} \tag{7.4}$$

The junction condition that $U$ is smooth across the boundaries requires that the function $f(u)$ must have the same form in both regions II and IV. Similarly $g(v)$ must have the same form in regions III and IV.

At this point it is convenient to concentrate on the initial boundaries between regions I and II, and between regions I and III. It is then possible to use the transformations (6.7) to put $M = 0$ in all three of these initial regions. Equations $(6.22b, c)$, which now apply only to regions III and II respectively, become

$$\begin{aligned} g'' &= \tfrac{1}{2} e^{U} g'^2 - \tfrac{1}{2} e^{-U} (V_v^2 \cosh^2 W + W_v^2 + 4\Phi_0^\circ \bar{\Phi}_0^\circ) \\ f'' &= \tfrac{1}{2} e^{U} f'^2 - \tfrac{1}{2} e^{-U} (V_u^2 \cosh^2 W + W_u^2 + 4\Phi_2^\circ \bar{\Phi}_2^\circ). \end{aligned} \tag{7.5}$$

These equations, together with (7.3), imply that $f$ and $g$ are monotonically decreasing functions for positive arguments, at least in regions II and III. The scale transformations (6.7) cannot reverse the directions of the parameters $u$ and $v$. Thus, even in region IV with $M$ non-zero, $f$ and $g$ must be monotonically decreasing functions.

It is always possible, therefore, to use the transformation (6.7) to express the functions $f$ and $g$ in the forms

$$f = \tfrac{1}{2} - (c_1 u)^{n_1} \Theta(u), \qquad g = \tfrac{1}{2} - (c_2 v)^{n_2} \Theta(v) \tag{7.6}$$

which apply globally. It is also possible to use further transformations to put $c_1 = 1$ and $c_2 = 1$. However, these particular constants describe a measure of the magnitudes of the approaching waves, and it is therefore often convenient to retain them without this further rescaling. For situations in which $M = 0$ in the inital regions I, II and III, the product $c_1 c_2$ has an invariant meaning as the amount of non-linearity in the interaction. However, if transformations are used to put $c_1 = 1$ and $c_2 = 1$, then this magnitude is absorbed into the metric component $e^{-M}$.

The expressions (7.6) may be of particular convenience, as they transparently demonstrate the technique by which an exact solution obtained in region IV in terms of these functions can be extrapolated back to regions III, II and I. However, this particular parametrization is not always

the most convenient, as will be demonstrated for example in the Bell–Szekeres solution to be described in Chapter 15.

It may also be noted that in the line element (6.20) the metric coefficient $e^{-U}$, according to (6.24), is given by $f(u)+g(v)$ where $f$ and $g$ have now both been shown to be decreasing functions from the value $\frac{1}{2}$. It is therefore inevitable that a singularity will develop as $f+g\to 0$. Whether this is a curvature singularity or simply a coordinate singularity will have to be considered in detail. An initial discussion of this topic is given in the next chapter.

In practice, it is not easy to find exact solutions in the interaction region for any specified initial conditions in which the metric functions in regions I, II and III are the initial data. Although this approach will be discussed later, most of the explicit exact solutions that will be described in the following chapters have been obtained by first solving the field equations in region IV, and then extrapolating back to determine the approaching waves that would give rise to them. In this context it may be noted that, since $f$ and $g$ are monotonically decreasing functions, they may be adopted as coordinates in the interaction region. Also, since these functions are required to be smooth across the boundaries, they may easily be matched to the null coordinates $u$ and $v$ in regions II and III.

For colliding gravitational waves, attention is concentrated on equations $(6.22d,e)$. In the interaction region it is possible to use $f$ and $g$ as coordinates and these equations may then be integrated to give $V(f,g)$ and $W(f,g)$ subject to the initial data given by $V(\frac{1}{2},g)$ and $W(\frac{1}{2},g)$ on the surface $u=0$ and by $V(f,\frac{1}{2})$ and $W(f,\frac{1}{2})$ on the surface $v=0$. The situation for colliding electromagnetic waves is a little more complicated with the addition of (6.21) as well as the extra terms in $(6.22d,e)$. However, even in this case the junction conditions for $V$ and $W$ do not usually impose additional complications.

Once $V$ and $W$ are found from equations $(6.22d,e)$, it is then necessary to integrate $(6.22b,c)$ to obtain $M$. Because the integrability conditions are automatically satisfied, such a solution is known to exist.

Equations $(6.22b,c)$ can now be written in the form

$$\begin{aligned} M_v &= -\frac{g''}{g'}+\frac{g'}{2(f+g)}-\frac{(f+g)}{2g'}(V_v^2\cosh^2W+W_v^2+4\Phi_0^\circ\bar\Phi_0^\circ)\\ M_u &= -\frac{f''}{f'}+\frac{f'}{2(f+g)}-\frac{(f+g)}{2f'}(V_u^2\cosh^2W+W_u^2+4\Phi_2^\circ\bar\Phi_2^\circ). \end{aligned} \tag{7.7}$$

It is thus convenient to put

$$e^{-M}=\frac{f'g'}{\sqrt{f+g}}\,e^{-S} \tag{7.8}$$

where $S$ satisfies

$$\begin{aligned} S_g &= -\tfrac{1}{2}(f+g)\left(V_g^2\cosh^2 W + W_g^2 + 4\frac{\Phi_0^\circ\bar{\Phi}_0^\circ}{g'^2}\right)\\ S_f &= -\tfrac{1}{2}(f+g)\left(V_f^2\cosh^2 W + W_f^2 + 4\frac{\Phi_2^\circ\bar{\Phi}_2^\circ}{f'^2}\right). \end{aligned} \tag{7.9}$$

The boundary conditions discussed above require that $e^{-M}$ be continuous, and that $f(0)=\frac{1}{2}$, $f'(0)=0$, $g(0)=\frac{1}{2}$ and $g'(0)=0$. However, in view of (7.8), $e^{-M}$ cannot be continuous across $u=0$ and $v=0$ unless $e^{-S}$ is unbounded on these boundaries. For the junction conditions to be satisfied, it is therefore essential that the functions $V$, $W$, $\Phi_0^\circ$ and $\Phi_2^\circ$, which are solutions of equations (6.22$d,e$) and (6.21), should be such that the solution of (7.9) for $S$ must contain terms of the form

$$S = k_1\log(\tfrac{1}{2}-f) + k_2\log(\tfrac{1}{2}-g) + \log(p_1p_2) + \dots \tag{7.10}$$

where $p_1$ and $p_2$ are constants. If the leading terms in a power series expansion for $f$ and $g$ in region IV have the form

$$f = \tfrac{1}{2} - (c_1u)^{n_1} + \dots, \qquad g = \tfrac{1}{2} - (c_2v)^{n_2} + \dots \tag{7.11}$$

then $e^{-M}$ is continuous across the boundaries only if $S$ contains the terms in (7.10) where the constants $k_1$ and $k_2$ are given by

$$k_1 = 1 - 1/n_1, \qquad k_2 = 1 - 1/n_2. \tag{7.12}$$

In order to satisfy (7.3), it is necessary that $n_1 \geq 2$ and $n_2 \geq 2$. It therefore follows that $k_1$ and $k_2$ must be restricted to the range

$$\tfrac{1}{2} \leq k_1, k_2 < 1. \tag{7.13}$$

In addition, in order to put $M$ zero in region I, it is also appropriate to put

$$p_1 = -n_1c_1, \qquad p_2 = -n_2c_2. \tag{7.14}$$

It can be seen that the boundary conditions that are appropriate for colliding plane waves become fairly complicated. It is mainly a question of choosing solutions $V$, $W$, $\Phi_0^\circ$ and $\Phi_2^\circ$ of equations (6.22$d,e$) and (6.21) such that the solution $M$ of (6.22$b,c$) contains terms of the form (7.10). Even then, the restrictions (7.12) are placed on the parameters and hence on the forms of $f$ and $g$.

Since the main field equations involve the functions $V$, $W$, $\Phi_0^\circ$ and $\Phi_2^\circ$, it is convenient to consider a form of the boundary conditions that applies to these functions only. This can be achieved by substituting (7.10) into (7.9) and considering boundaries as $f \to 1/2$ and $g \to 1/2$. The resulting conditions can be conveniently expressed in the form

$$\begin{aligned} \lim_{g\to 1/2} \left[ (\tfrac{1}{2} - g) \left( V_g^2 \cosh^2 W + W_g^2 + 4\frac{\Phi_0^\circ \bar{\Phi}_0^\circ}{g'^2} \right) \right] &= 2k_2 \\ \lim_{f\to 1/2} \left[ (\tfrac{1}{2} - f) \left( V_f^2 \cosh^2 W + W_f^2 + 4\frac{\Phi_2^\circ \bar{\Phi}_2^\circ}{f'^2} \right) \right] &= 2k_1 \end{aligned} \tag{7.15}$$

where $k_1$ and $k_2$ must satisfy (7.13). It is sometimes more convenient to revert to expressions involving the null coordinates $u$ and $v$. These become

$$\begin{aligned} \lim_{v\to 0} \left[ \frac{V_v^2 \cosh^2 W + W_v^2 + 4\Phi_0^\circ \bar{\Phi}_0^\circ}{v^{n_2-2}} \right] &= 2n_2(n_2-1)c_2^{n_2} \\ \lim_{u\to 0} \left[ \frac{V_u^2 \cosh^2 W + W_u^2 + 4\Phi_2^\circ \bar{\Phi}_2^\circ}{u^{n_1-2}} \right] &= 2n_1(n_1-1)c_1^{n_1}. \end{aligned} \tag{7.16}$$

This form of the conditions is particularly convenient when $n_1 = n_2 = 2$.

These conditions will be discussed further for colliding gravitational waves in Section 11.2 and for the collision of a mixture of electromagnetic and gravitational waves in Section 16.1.

Before concluding this chapter, it is appropriate to consider the possibilities that arise when the continuity conditions for $U$ are slightly relaxed, requiring only that it is continuous so that $n_1, n_2 \geq 1$. In this case, impulsive components occur in the Ricci tensor on the boundaries between the different regions. It is then possible that such components could be interpreted in terms of impulsive null matter fields. Explicit solutions of this type have in fact been given by Dray and 't Hooft (1986) and Tsoubelis (1989) and will be discussed in Section 20.4. However, the physical interpretation of such situations requires careful analysis. At this point, it may simply be pointed out that, in order for the matter tensor to have positive energy density, $f'$ and $g'$ must be negative on the boundaries.